International Journal of Theoretical Physics, Vol. 39, No. 3, 2000

The Order Topology on a Lattice and Its MacNeille Completion†

Vladimír Olejček[1]

Received December 8, 1999

An example shows that the order topology on a lattice can be strictly stronger than the order topology inherited from its MacNeille completion

1. INTRODUCTION

A net $\{x_\alpha\}_{\alpha \in \mathscr{E}}\}$ in a partially ordered set $(P, \preccurlyeq)$ is said to be *order convergent* briefly (o)-convergent (Birkhoff, 1973), to a point $x \in P(x_\alpha \overset{(2]}{\rightarrow} x)$ iff there are two nets, an increasing net $\{u_\alpha\}_{\alpha \in \mathscr{E}}$ and a decreasing net $\{u_\alpha\}_{\alpha \in \mathscr{E}}$, such that $u_\alpha \preccurlyeq x_\alpha \preccurlyeq v_\alpha$ for every $\alpha \in \varepsilon$ and

$$\bigvee_{\alpha \in \mathscr{E}} u_\alpha = \bigwedge_{\alpha \in \mathscr{E}} u_\alpha = x$$

The *order topology* τ_o on $(P, \preccurlyeq)$ is defined in Birkhoff (1973) by order convergence as follows: A set $C \subset P$ is said to be closed iff it contains the limits of all order-convergent nets of elements of C. It is the finest topology on P which preserves the order convergence. It follows that every order-convergent net in P is convergent with respect to the order topology $(x_\alpha \overset{\tau_o}{\rightarrow} x)$.

If the MacNeille completion $\hat{P}$ of P is taken into account, then, in addition, two more kinds of convergence can be considered on P, namely, the order convergence inherited from the order convergence on $\hat{P}$ and the convergence with respect to the order topology inherited from $\hat{P}$. In Birkhoff (1973, Chap. X, par. 9), the author makes no distinction among the above

†This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

[1] Departement of Mathematics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, SK-81219 Bratislava, Slovakia.

four kinds of convergence. Kirchheimová (1990) gave an example of a lattice in which the order convergence in P does not coincide with the order convergence on $\hat{P}$. Riečanová then raised the question of characterizing partially ordered structures in which the various pairs of these four convergences coincide. Some partial answers follow from Riečanová (1993, 1998), and Erné (1980), and some examples are presented in Olejček (n.d.). In particular, an example shows that in general the order topology τ_o on a poset P can be strictly stronger than the topology $\hat{\tau}_o \cap P$ inherited from $\hat{P}$. In this note we show that the same is true even if P is a lattice.

2. THE MAIN RESULT

Example. Let us consider sequences $C = \{c(i)\}$, $D = \{d(j)\}$, and double indexed sequences $A = \{a(i, j)\}$, $B = \{b(i, j)\}$, where $i \in \{\ldots, -2, -1, 1, 2, \ldots\}$, $j \in \{1, 2, \ldots\}$, and we have elements e, 0, 1. The set $L = A \cup B \cup C \cup \{e\} \cup \{0\} \cup \{1\}$ equipped with the order defined by the Hasse diagram in Fig. 1 (the first three elements of each sequence are described only) is a lattice. The set B is a τ_o-closed set in L since it contains no infinite (o)-convergent nets. The MacNeille completion of L is the complete lattice

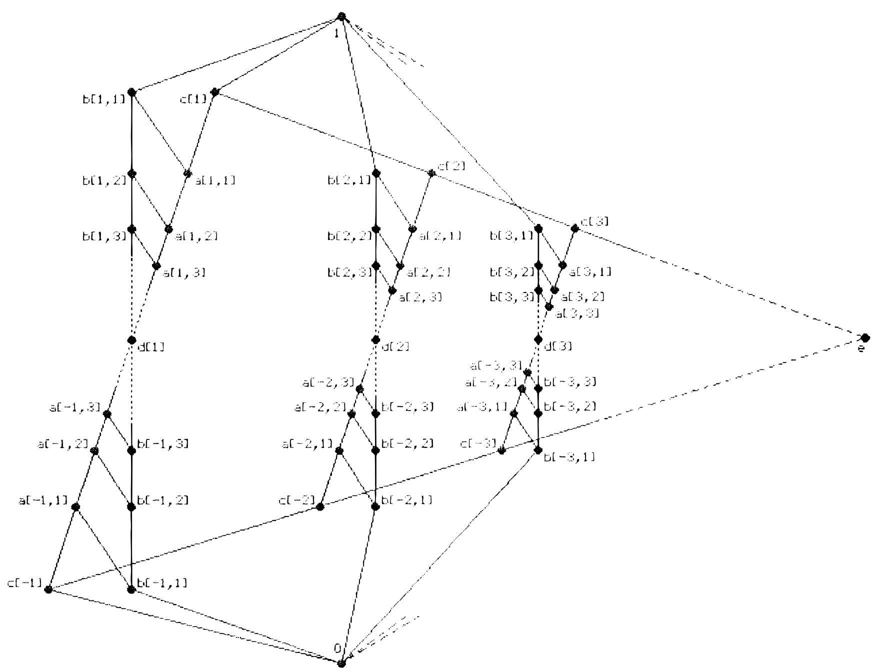

Fig. 1.

$\hat{L} = L \cup D$. It follows that the closure $\overline{B}$ of B with respect to the order topology τ_o on $\hat{L}$ contains not only all the points of the set D, which do not belong to L, but also the point e, which is the $(\hat{o})$-convergence limit of $d(j)$, $j = 1, 2, \ldots$, in $\hat{L}$ and therefore also belongs to the closure of B with respect to the topology $\hat{\tau}_o \cap L$. Thus B is not closed with respect to the topology $\hat{\tau}_o \cap L$ and consequently τ_o is strictly stronger than $\hat{\tau}_o \cap L$.

ACKNOWLEDGMENT

This work was supported by Grant 1/7625/20 MŠ SR.

REFERENCES

Birkhoff, G. (1973), *Lattice Theory*, 3rd ed., American Mathematical Society, Providence, Rhode Island.

Erné, M. (1980), Order-topological lattices, *Glasgow Math. J.* **21**, 57–68.

Kirchheimová, H. (1990), Some remarks on (o)-convergence, in *Proceedings of the First Winter School of Measure Theory in Liptovský Ján*, pp. 110–113.

Riečanová, Z. (1993), Topological and order-topological orthomodular lattices, *Bull. Aust. Math. Soc.* **17**, 509–518.

Riečanová, Z. (1998). Strongly compactly atomistic orthomodular lattices and modular ortholattices, *Tatra Mt. Math. Publ.* **15**, 143–153.

Olejček, V. Order convergence and order topology on a poset, *Int. J. Theor. Phys.*, **38**, 552–562.